Homework 1

- **Section 1.1, Q 12**: In this question, we want to list the elements of the set

  $$\{x \in \mathbb{Z} : |2x| < 5\}.$$

  First of all let's try to understand what this notation means. By the definition of the set notation, this means, "the set of all x's in the set of integers such that the absolute value of $2x$ is less than 5". But then the elements of this set become more clear. We see that we need to find all the integers x such that $-5 < 2x < 5$, i.e. $-\frac{5}{2} < x < \frac{5}{2}$, and those numbers are $-2, -1, 0, 1, 2$.

  Thus the set is $\{-2, -1, 0, 1, 2\}$.

- **Section 1.1, Q 14**: This question looks like the previous question with a little twist in the expression of the numbers. Again, all we need to do is to be a little careful in how we read the set builder notation. The set given is

  $$\{5x : x \in \mathbb{Z}, |2x| \leq 8\}.$$

  What this set builder notation says is "the set of all $5x$'s where x is in the set of integers and the absolute value of $2x$ is less than or equal to 8 ". So, to find all the $5x$'s we need to find all the x's. From the definition of the set, we see that x is an integer and $|2x| \leq 8$, i.e. $-8 \leq 2x \leq 8$, i.e. $-4 \leq x \leq 4$. Thus, all the possible x's are "$-4, -3, -2, -1, 0, 1, 2, 3, 4$". We also know that the set is the set of all $5x$'s where x's are as found.

  Hence the set is $\{-20, -15, -10, -5, 0, 5, 10, 15, 20\}$.

- **Section 1.1, Q 24**: In this question, we want to write the set

  $$\{-4, -3, -2, -1, 0, 1, 2\},$$

  in the set builder notation. There are more than one way to write this set in set builder notation. One way to do it is to see that these numbers are all integers and that they are successive (the way the elements are listed is helping us in this case). We can combine these observations to realize that these numbers are all the integers between -4 and 2. Now all we have to do is to wite this down in set builder notation.

  The set can be written as $\{x \in \mathbb{Z} : -4 \leq x \leq 2\} = \{x : x \in \mathbb{Z}, -4 \leq x \leq 2\} = \{x \in \mathbb{Z} : |x + 1| \leq 3\}$, or in many other ways.

- **Section 1.1, Q 26**: In this question, we want to write the set

  $$\{\ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \ldots\},$$

  in the set builder notation. For this question, we need to find the common expression and the rule the elements follow. The apparent relation we see is that the numbers

"$3, 9, 27$" are the natural number powers of 3 (and 1 is the zeroth power). We also remember that when we have "one over a number" than this means "taking a negative power". Thus, this suggests that te numbers $\frac{1}{27}, \frac{1}{9}, \frac{1}{3}$ are also the powers of 3, but only the negative integer powers. This suggests that the set is the set of all integer powers of 3. If we write this in set builder notation, we get,

$\{3^i : i \in \mathbb{Z}\}$.

- **Section 1.1, Q 32**: In this question, we want to find the cardinality of the set $\{\{\{1,4\}, a, b, \{\{3,4\}\}, \{\emptyset\}\}\}$. Here, all we need to do is to be careful with the curly brackets. We see that this is the set (reading the brackets from outermost to innermost), that consists of the set $\{\{1,4\}, a, b, \{\{3,4\}\}, \{\emptyset\}\}$.

  Although this set $\{\{1,4\}, a, b, \{\{3,4\}\}, \{\emptyset\}\}$, which is the only element of the given set, has 5 elements: a set that contains 1 and 4, a, b, a set that consists of another set that contains 3 and 4, and a set that consists of the empty set, the original set has only one element, the set $\{\{1,4\}, a, b, \{\{3,4\}\}, \{\emptyset\}\}$.

  Thus, since the size of a finite set is the number of elements of the set, we get,

  $|\{\{\{1,4\}, a, b, \{\{3,4\}\}, \{\emptyset\}\}\}| = 1$.

- **Section 1.2, Q 2(a)**: In this question, we want to find $A \times B$ where $A = \{1, 2, 3, 4\}$ and $B = \{a, c\}$.

  By definition, $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

  Thus, $A \times B = \{(1, a), (2, a), (3, a), (4, a), (1, c), (2, c), (3, c), (4, c)\}$.

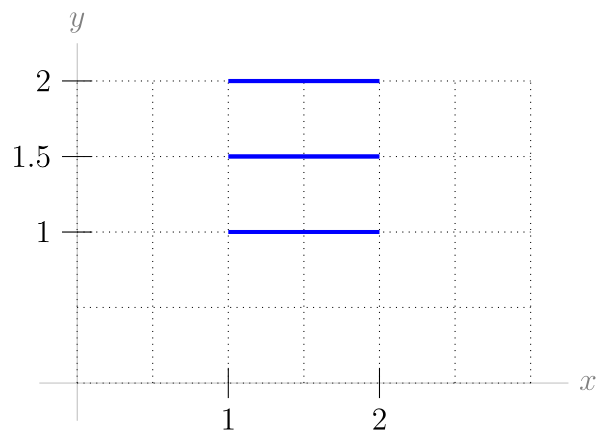
- **Section 1.2, Q 2(e)**: In this question, we want to find $A \times \emptyset$ where $A = \{\pi, e, 0\}$. Again, by definition, $A \times \emptyset$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in \emptyset$. But this time, we see that since $\emptyset$ has no elements, we cannot have any such ordered pairs.

  Thus, $A \times \emptyset = \emptyset$.

- **Section 1.2, Q 6**: In this question, we want to find $A \times B$ where $A = \{x \in \mathbb{R} : x^2 = x\}$ and $B = \{x \in \mathbb{N} : x^2 = x\}$. To find the cartesian product of A and B, we need to find what A and B are. We see that A is the set of all real numbers whose square is itself, and we know those numbers are 0 and 1. Hence, $A = \{0, 1\}$. The set B looks almost exactly same as the set A with the difference that the numbers x's are from the natural numbers. Thus, $B = \{1\}$.

  Therefore, $A \times B = \{(a, b) : a \in A, b \in B\} = \{(0, 1), (1, 1)\}$.

- **Section 1.2, Q 14**: In this question, we want to sketch $A \times B$ where $A = [1, 2]$ and $B = \{1, 1.5, 2\}$ on the xy plane. We see from the definition of the cartesian products, $A \times B$ is the set of all ordered pairs (in this case points in the plane) where the first coordinate is from A and the second coordinate is from the set B. This means that the x coordinate is chosen from the $A = [1, 2]$ where the y coordinate is either 1, 1.5, or 2. Therefore, if we sketch this set on the plane, we get,



- **Section 1.3, Q 8**: In this question, we want to find all the subsets of the set $\{\{0, 1\}, \{0, 1, \{2\}\}, \{0\}\}$. For this, we need to find all the elements of this set. We see that the elements of this set are $\{0, 1\}$, $\{0, 1, \{2\}\}$, $\{0\}$, i.e. it has 3 elements. This means that we need to find $2^3 = 8$ subsets and they are:

  $\emptyset$ (as always), $\{\{0, 1\}, \{0, 1, \{2\}\}, \{0\}\}$ (i.e. the set itself, as always), $\{\{0, 1\}\}$, $\{\{0, 1, \{2\}\}\}$, $\{\{0\}\}$, $\{\{0\}, \{0, 1, \{2\}\}\}$, $\{\{0, 1\}, \{0, 1, \{2\}\}\}$, and $\{\{0\}, \{0, 1\}\}$.

- **Section 1.3, Q 10**: In this question, we need to list the elements of the set, $\{X \subseteq \mathbb{N} : |X| \leq 1\}$. First, we need to understand what this set is. This set is the set of all subsets of the set of natural numbers whose size is less than or equal to 1. First we see that there is only one set whose size ls less than 1, i.e. 0, and that is the empty set.

  Now, we have to find all the subsets of the set of natural numbers whose size is 1, i.e. who has only one element. But, then, this is also quite easy, these are just the sets of the form $\{n\}$, where $n \in \mathbb{N}$.

  Thus, $\{X \subseteq \mathbb{N} : |X| \leq 1\} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \ldots\}$.

- **Section 1.4, Q 2**: In this question, we need to find the set $\mathcal{P}(\{1, 2, 3, 4\})$. By definition, we know that this is the power set of the set $\{1, 2, 3, 4\}$, which is the set of all subsets of the set $\{1, 2, 3, 4\}$.

  Thus,

$$\begin{aligned}\mathcal{P}(\{1, 2, 3, 4\}) = \{&\emptyset, \{1, 2, 3, 4\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\},\\ &\{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\{1, 3, 4\}, \{2, 3, 4\}\}.\end{aligned}$$

- **Section 1.4, Q 6**: In this question, we want to find $\mathcal{P}(\{1, 2\}) \times \mathcal{P}(\{3\})$. To find this cartesian product, we need to find the sets $\mathcal{P}(\{1, 2\})$ and $\mathcal{P}(\{3\})$ first.

By definition, $\mathcal{P}(\{1,2\})$ is the set of all subsets of the set $\{1,2\}$. Thus, $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1,2\}, \{1\}, \{2\}\}$. Similarly, $\mathcal{P}(\{3\}) = \{\emptyset, \{3\}\}$.

Now, using the definition of the cartesian product, we see, $\mathcal{P}(\{1,2\}) \times \mathcal{P}(\{3\}) = \{(\emptyset,\emptyset), (\emptyset,\{3\}), (\{1,2\},\emptyset), (\{1,2\},\{3\}), (\{1\},\emptyset), (\{1\},\{3\}), (\{2\},\emptyset), (\{2\},\{3\})\}$.

- **Section 1.4, Q 20**: In this question, we want to find $|\{X \subseteq \mathcal{P}(A) : |X| \leq 1\}|$ for a set A of cardinality m, i.e. the cardinality of the set of all subsets of the set of subsets of the set A of size less than or equal to 1. Although it sounds complicated, all we have to do is to be a little bit more careful.

  First of all, if we have a set A of size m, then we know that the size of $\mathcal{P}(A)$ is 2^m, meaning that $\mathcal{P}(A)$ has 2^m elements, two of them being the empty set and the set A itself. Now, if we go back to the question, we see that all we need to find is the cardinality of the set of all subsets of a set of size 2^m which has size less than or equal to 1. As in Question 10 of section 1.3, we see that there is only one subset of size 0 which is the empty set. Thus $\emptyset$ is in the set $\{X \subseteq \mathcal{P}(A) : |X| \leq 1\}$. Now we also have to find all the subsets of $\mathcal{P}(A)$ of size 1. Similar to Question 10 of section 1.3, all the subsets of size one are the sets that only contain one element of the set $\mathcal{P}(A)$. Since $\mathcal{P}(A)$ has 2^m elements, there can be 2^m such subsets (two of those subsets being $\{\emptyset\}$ and $\{A\}$).

  Therefore, $|\{X \subseteq \mathcal{P}(A) : |X| \leq 1\}| = 2^m + 1$, where "+1" comes from the empty set.